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Traveling toward Proof

Student success with proof is an important but elusive goal of many mathematics programs (Senk 1985; Battista and Clements 1995). After lengthy struggles with this challenge, we have invented an extended metaphor that happily bridges the gap between students' work with logic and their creation of proofs. It fits into our curriculum after students have been well immersed in multiple translations of conditionals and are familiar with valid arguments, in particular, the direct argument, or modus ponens, and the chain rule, or law of transitivity.

The metaphor uses an imaginary airline that is named after a famous mathematician or logician (Rubenstein, Craine, Butts, et al. 1998, 394). We assume that Aristotle Airlines serves cities in North America with the routes shown in the route map above. We ask students to think of themselves as travel agents who plan itineraries for air travel. For planning purposes, we can write a flow diagram with arrows connecting cities and with flights indicated under each arrow. For example, the following diagram could represent an Aristotle Airlines itinerary for travel from New York to Mexico City.

New York $\xrightarrow[\text{Flight 108}]{}$ **Chicago** $\xrightarrow[\text{Flight 105}]{}$ **Mexico City**

This format leads quite naturally to flow proofs, where the cities correspond to statements, the arrows show conditionals, and the flights represent reasons associated with each arrow.

The itinerary could also be represented in a chart with cities and flights:

City	Flight
New York	Starting point
Chicago	Flight 108: New York → Chicago
Mexico City	Flight 105: Chicago → Mexico City

This airline-ticket format builds very nicely toward students' writing of two-column proofs. The cities correspond to statements in two-column-proof format, and the flights correspond to reasons. The starting point, of course, corresponds to the reason "given." We use the arrow notation for flights because it is familiar to students who have already been introduced to it for conditionals in a previous logic unit. Flight 108, for example, can be read, "If I am in New York, then I can go to Chicago." Students readily assimilate many analogies between the itinerary-writing process and the construction of proofs, including the following:

- Travel must originate at the starting city (the given information) and arrive at the ultimate destination (what is to be proved).
- A city (a statement) that links two points cannot be skipped.
- A flight (a reason) must join any two cities that you claim to connect.

Besides these correspondences, another analogy with mathematical proof is that more than one way may exist to get from the starting point, or the hypothesis, to the destination, or what is to be proved. For instance, the reader can find two different itineraries from Montreal to Chicago. Finding more than one itinerary for a trip helps students appreciate that more than one correct proof is possible for an assertion. Serendipitously, in later work with proofs, many phrases that apply to travel apply as well to students' proof-writing efforts:

More than one way may exist to get from the starting point to the destination

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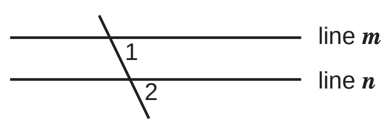
- ✈ "You got us to Chicago, but we were heading for New York," for "proving" something other than what was asked.
- ✈ "You took us by a long route, but you got us there," for using unnecessary steps.
- ✈ "You gave us a short flight," for a particularly compact proof.

You got us to Chicago, but we were heading for New York!

As the astute reader has undoubtedly noticed, flying with Aristotle Airlines clearly has drawbacks. Since its flights are all one-way, the traveler must be careful to avoid traveling in the wrong direction. We consider, for instance, this proposed itinerary for a flight from Los Angeles to New York. Students are asked to find the error.

City	Flight
Los Angeles	Starting point
Chicago	Flight 110: L.A. → Chicago
New York	Flight 108: New York → Chicago

The problem is that flight 108 originates in New York; it does not terminate there. This situation is essentially an instance of a converse error! Similarly, students need to be reminded in using reasons for steps in a proof that the "hypothesis" in the conditional must be satisfied to deduce the conclusion. This situation corresponds to a common mistake that students make when using conditionals in a two-column proof, as shown in **figure 1.** Helping students avoid this error in proof writing is much easier, since we can refer to the difficulty in terms of the travel analogy.



1. line m ‖ line n	1. Given
2. $\angle 1 \cong \angle 2$	2. If a transversal intersects two lines so that corresponding angles are congruent, then the lines are parallel.

Fig. 1
The converse error in a geometric proof

The airline analogy may be carried one step further by introducing two-way flights, which correspond to biconditional statements, that is, if-and-only-if statements. We suppose that Aristotle Airlines decides to expand its service to Vancouver by adding a round-trip connection through Los Angeles. This situation can be indicated with a double arrow, as follows:

Flight 201: Vancouver ↔ Los Angeles

An advantage of two-way flights (aside from their obvious convenience—one wonders how Aristotle Airlines could survive in a competitive market without them!) is that the cities may appear in the itinerary in either order. For instance, the sequences

City	Flight
Los Angeles	Given
Vancouver	Flight 201: L.A. ↔ Vancouver

and

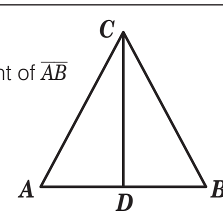
City	Flight
Vancouver	Given
Los Angeles	Flight 201: Vancouver ↔ L.A.

are both possible.

In a geometric proof, two-way flights, or biconditionals, occur most frequently in the form of definitions. We have found that by thinking of definitions as two-way flights, students can appreciate why they work "in both directions" in a proof. In contrast, most postulates and theorems are one-way conditionals, and students must be careful with them to avoid the converse error.

As with all analogies, our airline-ticket–proof analogy has limitations. An important one arises when students encounter proofs involving conditionals with multiple antecedents, for example, in proving triangles congruent using side-side-side. (See **fig. 2.**) Three conditions must be met before drawing the conclusion. To help both constructors and readers of proofs follow good reasoning, braces can be used in flow diagrams, and reference num-

Given: D is the midpoint of $\overline{AB}$
$\overline{AC} \cong \overline{BC}$
Prove: $\triangle ADC \cong \triangle BDC$.



$$D \text{ is the midpoint of } \overline{AB} \xrightarrow{\text{(def. of midpoint)}} \left.\begin{array}{l} \overline{AD} \cong \overline{BD} \\ \overline{AC} \cong \overline{BC} \text{ (given)} \\ \overline{CD} \cong \overline{DC} \text{ (reflexive property)} \end{array}\right\} \longrightarrow \triangle ADC \cong \triangle BDC \text{ (SSS)}$$

(a)

Statements	Reasons
1. D is the midpoint of $\overline{AB}$	1. Given
2. $\overline{AD} \cong \overline{BD}$	2. M is the midpoint of $\overline{XY}$ $\Leftrightarrow$ M is on $\overline{XY}$ and $\overline{XM} \cong \overline{MY}$ (1)
3. $\overline{DC} \cong \overline{DC}$	3. Reflexive property
4. $\overline{AC} \cong \overline{BC}$	4. Given
5. $\triangle ADC \cong \triangle BDC$	5. SSS $\Rightarrow \cong \triangle$s (2, 3, 4)

(b)

Fig. 2
Two forms of a proof using an implication with multiple antecedents

bers can be attached to reasons in a two-column format (Hirsch et al. 1990, 96). Reference numbers point to previous statements in the proof that are hypotheses for a conditional used as a reason, as in **figure 2.** In the spirit of Aristotle Airlines, the reference number tells "where you are coming from."

CONCLUSION

We have found that the extended metaphor of Aristotle Airlines is extremely helpful in supporting students' construction of proofs. It fits well with both flow and two-column forms, and it acts as a stepping-stone to paragraph proofs, as well. See Brandell (1994) for more on paragraph proofs. In our experience, students who encounter this useful analogy make more sense of proofs and proof-writing and do so with success and enthusiasm.

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